1. CONGRUENCE OF TRIANGLES

Two Δs are congruent, if there exists a correspondence between their vertices such that the corresponding sides and the corresponding angles of the two Δs are equal or congruent.

1.1 CONGRUENCE RELATION

(i) Every Δ is congruent to itself i.e. $\Delta ABC \cong \Delta ABC$

(ii) If $\Delta ABC \cong \Delta DEF$
then $\Delta DEF \cong \Delta ABC$

(iii) If $\Delta DEF \cong \Delta ABC$, and
then $\Delta DEF \cong \Delta PQR$ then
$\Delta ABC \cong \Delta PQR$

1.2 CONGRUENCE OF TRIANGLES

1.2.1 Side-angle-side congruence criterion

Theorem 1 : Two triangles are congruent if two sides and the included angle of one are equal to the corresponding sides and the included angle of the other triangle.

2. ANGLE-SIDE-ANGLE (ASA) CONGRUENCE CRITERION

Theorem 2 : Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

Given : Two Δ*s ABC* and *DEF* such that $\angle B = \angle E$, $\angle C = \angle F$ and $BC = EF$

To prove : $\Delta ABC \cong \Delta DEF$

Proof : There are three possibilities

Case I : When $AB = DE$:

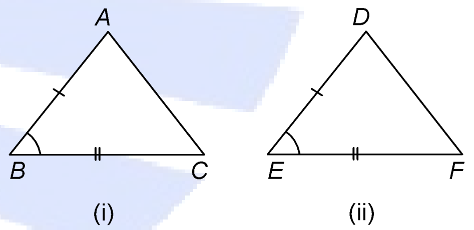
In this case, we have

$AB = DE$

$\angle B = \angle E$

and, $BC = EF$

So, by SAS criterion of congruence, we have



$\Delta ABC \cong \Delta DEF$

Case II : When $BA < ED$

In this case take a point *G* on *ED* such that $EG = BA$. Join *GF*.

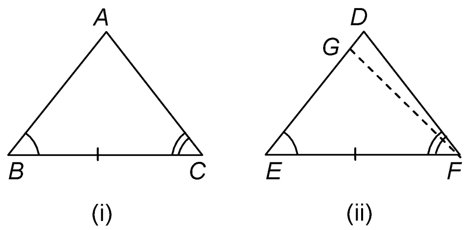
Now, in Δ*s ABC* and *GEF*, we have

$BA = GE$ [Assumed]

$\angle B = \angle E$ [Given]

and, $BC = EF$ [Given]

So, by SAS criterion of congruence, we have



$\Delta ABC \cong \Delta GEF$

$\Rightarrow$ $\angle ACB = \angle GFE$ [cpct]

But, $\angle ACB = \angle DFE$ [Given]

$\therefore$ $\angle GFE = \angle DFE$

This is possible only when ray FG coincides with ray FD or G coincides with D.

Therefore, BA must be equal to ED.

Thus, in Δs ABC and DEF, we have

$AB = DE$ [As proved above]

$\angle B = \angle E$ [Given]

and, $BC = EF$ [Given]

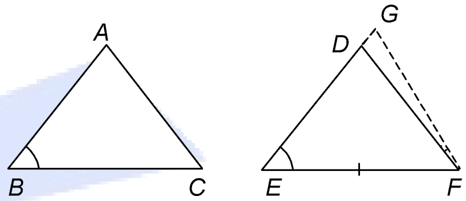
So, by SAS criterion of congruence, we have

$\Delta ABC \cong \Delta DEF$

Case III : when $BA > ED$

In this case take a point G on ED produced such that $EG = BA$. Join GF. Now, proceeding as in case II, we prove that

$\Delta ABC \cong \Delta DEF$



(i) (ii)

Hence, in all the three case, we have $\Delta ABC \cong \Delta DEF$

3. ANGLE- ANGLE-SIDE (AAS) CONGRUENCE CRITERION

Theorem 3 : If any two angles and a non-included side of one triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

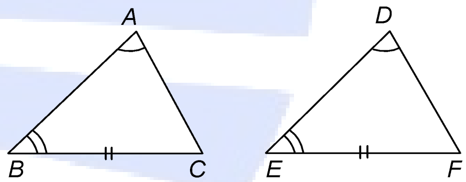
Given : Two Δs ABC and DEF such that

$\angle A = \angle D$, $\angle B = \angle E$, $BC = EF$

To prove : $\Delta ABC \cong DEF$

Proof : We have,

$\angle A = \angle D$, and $\angle B = \angle E$



(i) (ii)

$\Rightarrow$ $\angle A + \angle B = \angle D + \angle E$

$\Rightarrow$ $180° - \angle C = 180° - \angle F$ [Angle sum property]

$\Rightarrow$ $\angle C = \angle F$ …(i)

Thus, in Δs ABC and DEF, we have

$\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$

Now, in ΔABC and ΔDEF, we have

$\angle B = \angle E$ [Given]

$BC = EF$ [Given]

and, $\angle C = \angle F$ [From (i)]

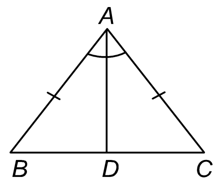
So, by ASA criterion of congruence, $\Delta ABC \cong \Delta DEF$

Theorem 4 : Angles opposite to equal sides of an isosceles triangle are equal.

Given : An isosceles triangle ABC in which $AB = AC$.

To prove : $\angle B = \angle C$.

Construction : Draw the bisector of $\angle A$ and let D be the point of intersection of this bisector of $\angle A$ and BC.



Proof : In ΔBAD and ΔCAD

$AB = AC$ [Given]

$\angle BAD = \angle CAD$ [By construction]

$AD = AD$ [Common]

$\Delta BAD \cong \Delta CAD$ [By SAS Congruency]

$\angle ABD = \angle ACD$ [cpct]

$\therefore \quad \angle B = \angle C$

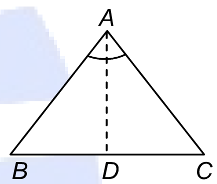
Theorem 5 : If two angles of a triangle are equal, then sides opposite to them are also equal.

Given : $A\ \Delta ABC$ in which $\angle B = \angle C$

To prove : $AC = AB$

Construction : Draw the bisector of $\angle A$ and let it meet BC at D.

Proof : In Δs ABD and ACD, we have



$\angle B = \angle C$ [Given]

$\angle BAD = \angle CAD$ [By construction]

$AD = AD$ [Common]

So, by SAS criterion of congruence, we have

$\Delta ABD \cong \Delta ACD$

$\Rightarrow \quad AB = AC$ [cpct]

4. SIDE-SIDE-SIDE(SSS) CONGRUENCE CRITERION

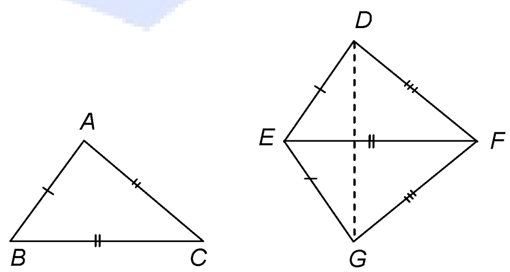
Theorem 6 : Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

Given : Two Δs ABC and DEF such that $AB = DE$, $BC = EF$ and $AC = DF$.

To prove : $\Delta ABC \cong \Delta DEF$

Construction : Suppose BC is the longest side. Draw EG such that $\angle FEG = \angle ABC$ and $EG = AB$. Join GF and GD.

Proof : In Δs ABC and ΔGEF, we have



(i) (ii)

$BC = EF$ [Given]

$AB = GE$ [By construction]

and, $\angle ABC = \angle FEG$ [By construction]

So, by SAS criterion of congruence, we have

$\Delta ABC \cong \Delta GEF$

⇒ $\angle A = \angle G$ and $AC = GF$ [c.p.c.t.]

Now, $AB = DE$ and $AB = GE$ …(i)

⇒ $DE = GE$

Similarly, $AC = DF$ and $AC = GF$ …(ii)

⇒ $DF = GF$

In ΔEGD, we have

$DE = GE$ [From (i)]

⇒ $\angle EDG = \angle EGD$ …(iii) [Angles opp. to equal side]

In ΔFGD, we have

$DF = GF$ [From (ii)]

⇒ $\angle FDG = \angle FGD$ …(iv) [Angles opp. to equal side]

From (iii) and (iv), we have

$\angle EDG + \angle FDG = \angle EGD + \angle FGD$

⇒ $\angle D = \angle G$

But, $\angle G = \angle A$ [Proved above]

∴ $\angle A = \angle D$ …(v)

Thus, in Δs ABC and DEF, we have

$AB = DE$ [Given]

$\angle A = \angle D$ [From (v)]

and, $AC = DF$ [Given]

So, by SAS criterion of congruence, we have

$\Delta ABC \cong \Delta DEF$

5. RIGHT ANGLE-HYPOTENUSE-SIDE (RHS) CONGRUENCE CRITERION

Theorem 7 : Two right triangles are congruent if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other triangle.

Given : Two right angles ABC and DEF in which $\angle B = \angle E = 90°$, $AC = DF$, $BC = EF$

To prove : $\Delta ABC \cong \Delta DEF$

Construction : Produce DE to G so that $EG = AB$. Join GF

Proof : In Δs ABC and ΔGEF, we have

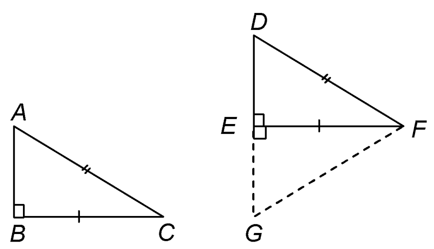
$AB = GE$ [By construction]

$\angle B = \angle FEG$ [Both 90º]

and, $BC = EF$ [Given]

So, by SAS criterion of congruence, we have

$\Delta ABC \cong \Delta GEF$



(i) (ii)

⇒ $\angle A = \angle G$ …(i)

$AC = GF$ [cpct] ...(ii)

Now, $AC = GF$ [From (ii)]

and, $AC = DF$ [Given]

∴ $DF = GF$

⇒ $\angle D = \angle G$ [Angles opposite to equal sides] ...(iii)

From (i) and (iii), we get

$\angle A = \angle D$ ...(iv)

Thus, in Δs *ABC* and *DEF*, we have

$\angle A = \angle D$ [From (iv)]

$\angle B = \angle E$ [Both 90º]

⇒ $\angle A + \angle B = \angle D + \angle E$

⇒ $180° - \angle C = 180° - F$ [Angle sum property]

⇒ $\angle C = \angle F$ ...(v)

Now, in Δs *ABC* and *DEF*, we have

$BC = EF$ [Given]

$\angle C = \angle F$ [From (v)]

and, $AC = DF$

So, by SAS criterion of congruence, we have

$\Delta ABC \cong \Delta DEF$

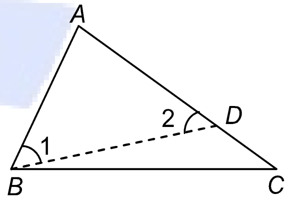
6. INEQUALITIES IN A TRIANGLE

Theorem 8: If two sides of a triangle are unequal, the longer side has greater angle opposite to it.

Given : A Δ *ABC* in which $AC > AB$

To prove : $\angle ABC > \angle ACB$

Construction : Mark a point *D* on *AC* such that *AB* = *AD*. Join *BD*.



Proof : In Δ*ABD*, we have

$AB = AD$ [By construction]

⇒ $\angle 1 = \angle 2$ [∵ Angles opp. to equal sides are equal] ...(i)

$\angle 2$ is the exterior angle of Δ*BCD* and an exterior angle is always greater than interior opposite angle. Therefore,

$\angle 2 > \angle DCB$

⇒ $\angle 2 > \angle ACB$ [∵ $\angle ACB = \angle DCB$] ...(ii)

From (i) and (ii), we have

⇒ $\angle 1 > \angle ACB$ ...(iii)

∴ $\angle ABC > \angle 1$ ...(iv)

From (iii) and (iv), we get

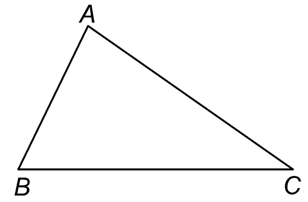
$\angle ABC > \angle ACB$

Theorem 9 : In a triangle the greater angle has the longer side opposite to it.

Given : A ΔABC in which $\angle ABC > \angle ACB$

To prove : $AC > AB$

Proof : In ΔABC, we have the following three possibilities



(i) $AC = AB$

(ii) $AC < AB$

(iii) $AC > AB$

Case I : When $AC = AB$

$AC = AB$

$\Rightarrow$ $\angle ABC = \angle ACB$ [Angle opp. to equal sides are equal]

This is a contradiction,

Since $\angle ABC > \angle ACB$ [Given]

$\therefore$ $AC \neq AB$

Case II : When $AC < AB$

if $\angle AC < AB$

$\Rightarrow$ $\angle ACB > \angle ABC$ [∵ Longer sides has the greater angle opposite to it]

This is also a contradiction

$\therefore$ We are left with the only possibility, $AC > AB$, which must be true

Hence, $AC > AB$

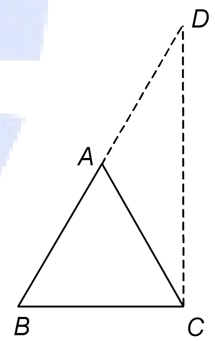
Theorem 10 : The sum of any two sides of a triangle is greater than the third side.

Given : A ΔABC

To prove : $AB + AC > BC$, $AB + BC > AC$ and $BC + AC > AB$

Construction : Produce side BA to D such that $AD = AC$. Join CD

Proof : In ΔACD, we have



$AC = AD$ [By construction]

$\Rightarrow$ $\angle ADC = \angle ACD$ [Angles opp. to equal sides are equal]

$\Rightarrow$ $\angle ACD = \angle ADC$

$\Rightarrow$ $\angle BCA + \angle ACD > \angle ADC$ [∵ $\angle BCA + \angle ACD > \angle ACD$]

$\Rightarrow$ $\angle BCD > \angle ADC$

$\Rightarrow$ $\angle BCD > \angle BDC$ [∵ $\angle ADC = \angle BDC$]

$\Rightarrow$ $BD > BC$ [∵ Side opp. to greater angle is larger]

$\Rightarrow$ $BA + AD > BC$

$\Rightarrow$ $BA + AC > BC$ [∵ $AC = AD$ (By construction)]

$\Rightarrow$ $AB + AC > BC$

Thus, $AB + AC > BC$

Similarly, $AB + BC > AC$ and $BC + AC > AB$